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Linear One-Step Processes with Artificial Boundaries

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An artificial absorbing boundary is introduced in a linear birth and death stochastic process in order to understand the long time behavior of an ecological community. The solution is obtained by means of a spectral resolution of the probability distribution. A more general linear process with a coefficient of arbitrary strength near the boundary both with absorbing and with reflecting boundary conditions is also studied.

KEY WORDS: One-step stochastic processes, spectral resolution, ecological community, artificial boundaries, neutral theory of biodiversity

1. INTRODUCTION

Birth and death stochastic processes have been studied in depth and applied to a wide variety of physical, chemical and biological systems (see Refs. 1 and 2). When the master equation governing these processes involves a boundary, one can solve it with artificial or natural boundary conditions, namely with or without altering the behavior of the birth and death coefficients near the edge (see Ref. 1). In other words, a master equation has an artificial boundary when there exists at least one site that is described by a special equation, which does not include the analytical expression of b_n (birth rate) or d_n (death rate) that governs the other sites.

For example, a quantized harmonic oscillator interacting with a radiation field can be described by a one-step master equation with $b_n = b(n + 1)$ and $d_n = dn$, which drives the probability per unit time for a jump from n to $n + 1$

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and from n to $n-1$, respectively. Without changing the value of the coefficients at the boundary $n=0$, it is possible to reach the well-known reflecting stationary solution $P_n \propto (b/d)^n$. In contrast, other phenomena described by $b_n = bn$ and $d_n = dn$ are naturally absorbing at $n=0$ and the trivial steady-state solution is $P_n = \delta_{n,0}$ (see Refs. 1 and 2).

Bounded random walks with reflecting and absorbing boundary conditions have been analyzed with coefficients of arbitrary strength at $n=0$ by involving some suitable equations at the boundary. One-step processes with these artificial boundaries have been extensively used in the study of many physical effects like evaporation of a gas through a surface, ionic currents through cell membranes, defect diffusion in crystals, chemical reactions and queuing problems (see Refs. 3–5).

Linear birth and death processes ($b_n = bn + \tilde{b}$, $d_n = dn + \tilde{d}$) have been studied by Karlin and McGregor Ref. 6 by using the left eigenvectors of the infinitesimal matrix generating the process. Their approach relies decisively on the knowledge of the elaborate structure of general birth and death processes as developed in Refs. 7 and 8. They demonstrated that a birth and death process is intimately linked to a Stieltjes moment problem. This connection enabled them to achieve the conditions for the existence and uniqueness of the spectral resolution of the probability distribution (see Ref. 9 for further developments along these lines).

In this paper we present a more straightforward and self-contained analysis of both the discrete spectrum and the (left and right) eigenvectors for linear one-step processes with artificial boundaries. Our approach relies only on some properties of the solution of the Laplace transformed equation of the process. Furthermore the spectral resolution of a more general problem (which includes both absorbing and reflecting boundaries) can also be obtained within our method.

In the last few years linear stochastic processes with artificial boundaries have been applied to the time evolution of ecosystems, mostly within the neutral theory of biodiversity (see Refs. 10–12). This theory provides a framework which accounts for the distribution of the relative species abundance both for the metacommunity and the local community (see Refs. 11 and 13).

The neutrality hypothesis is a symmetry assumption at the individual level. It postulates that all species obey the same interaction rules on a per capita basis (see Ref. 10). This oversimplified hypothesis, that resembles the ideal gas assumption in statistical physics, is equivalent to the assumption that the dynamics of the species is due to the similarity rather than differences between the species.

A slightly modified theory (see Ref. 12), which incorporates density dependence, assumes a rare species advantage by introducing a birth rate $b_n = bn + \tilde{b}$ and a death rate $d_n = dn + \tilde{d}$ for an arbitrary species with n individuals. Since a species disappears when its last individual dies, one should solve the master equation, which drives the population dynamics, with an artificial absorbing boundary at $n = 0$. We wish to stress that, even though in this case the stationary solution is trivial, owing to Frobenius' theorem (see Ref. 14), the first eigenfunction of the spectral resolution of $p_n(t)$ can be interpreted as the population distribution of a given species on time scales μ_1^{-1}, where μ_1 is the first eigenvalue. The continuum formulation with a Fokker–Planck equation for a less general diffusion problem was pointed out by Feller many years ago (see Ref. 15 he stressed the importance of such singular diffusion equations for the theory of stochastic processes and the theory of semigroups as well.

The structure of the paper is as follows: In Sec. 2 we will derive a solution for the generating function for the master equation with absorbing boundary conditions; In Sec. 3 we will find eigenvalues and eigenfunctions for the probability distribution function; and in Sec. 4 we will consider a solution for the generalized case of the boundary conditions.

2. ABSORBING CASE

In this section we present the calculations for the absorbing case. When dealing with absorbing boundaries, one should be aware of the existence and uniqueness of the solutions of the birth-death master equation. In fact, the non conservation of the total probability may cause a dependence on initial conditions of the stationary distribution. Yet if the process is linear, it is possible to prove both the existence and the uniqueness of the solution (see Ref. 16).

The linearity of the process also ensures that the spectrum is discrete with non negative eigenvalues (see below) when $b \neq d$, instead it is continuous and unbounded when $b = d$. If the process is a left-bounded random walk (symmetric or asymmetric), then the spectrum is continuous and bounded (see Refs. 1, 4 and 16).

Let us suppose that the time evolution of a population of a given species is governed by the following birth-death master equation ($n = 0, 1, 2, \ldots$) with *absorbing boundary conditions* (abc) at $n = 0$

$$\frac{\partial p_n(t)}{\partial t} = b_{n-1}p_{n-1}(t) + d_{n+1}p_{n+1}(t) - (b_n + d_n)p_n(t) \tag{1}$$

where the birth and death rates are given by

$$\begin{aligned} b_0 &= 0 \text{ and } \quad b_n = bn + \tilde{b} \quad \text{for} \quad n > 0 \\ d_0 &= 0 \text{ and } \quad d_n = dn + \tilde{d} \quad \text{for} \quad n > 0. \end{aligned} \tag{2}$$

To avoid ambiguities, during the calculations we shall suppose that $b, d > 0$ and $b \neq d$, but $\tilde{b}$ and $\tilde{d}$ could possibly take negative real values. Notice that if we know $p_1(t)$ then we also know $p_0(t)$, because $\dot{p}_0(t) = d_1 p_1(t)$ (we set $b_{-1} \equiv 0$).

We now seek an expansion in (right) eigenvectors of the solution of Eq. (1) in the abc case (2) with a specified *initial condition* $p_n(0)$. For our purposes, we define the following generating function

$$F(z,t) = z^{\varepsilon} \sum_{n=1}^{\infty} p_n(t) z^n \tag{3}$$

for $0 < z < 1$ and $0 < t < \infty$, where the real parameter ε will be defined later.

Using this definition we can transform the previous problem into an inhomogeneous first order p.d.e. for $F(z,t)$ with suitable initial conditions and choose the parameter ε to obtain a bounded solution. The p.d.e. is of the first order due to the linearity of the birth and death coefficients and it is inhomogeneous because of the barrier at $n = 0$. Carrying out the calculations, we get

$$\frac{\partial F(z,t)}{\partial t} = A(z)\frac{\partial F(z,t)}{\partial z} + \tilde{A}(z)F(z,t) + f(z,t) \tag{4}$$

where

$$A(z) = (1-z)(d-bz)$$

$$\tilde{A}(z) = (1-z)\frac{b\tilde{d} - d\tilde{b}}{d} \tag{5}$$

$$f(z,t) = -d_1 z^{\frac{\tilde{d}}{d}} p_1(t)$$

and $p_1(t)$ is the *unknown* probability of having just one individual at time t. Since n is not allowed to take negative values, we have fixed $\varepsilon = \tilde{d}/d$ to remove the singularity at $z = 0$. The initial condition for $F(z,t)$ is

$$F(z,0) = g(z) \tag{6}$$

where $g(z)$ is a sufficiently smooth function with $g(1) = 1$. This kind of inhomogeneous p.d.e. may be readily solved with the aid of the Duhamel's principle (see Refs. 17–19).

In fact, let $F_0(z,t)$ be the solution of the homogeneous p.d.e.

$$\begin{cases} \dfrac{\partial F_0(z,t)}{\partial t} = A(z)\dfrac{\partial F_0(z,t)}{\partial z} + \tilde{A}(z)F_0(z,t) \\ F_0(z,0) = g(z) \end{cases} \tag{7}$$

Moreover, let $F_1(z,t,\tau)$ be the solution of the following homogeneous p.d.e.

$$\begin{cases} \dfrac{\partial F_1(z,t,\tau)}{\partial t} = A(z)\dfrac{\partial F_1(z,t,\tau)}{\partial z} + \tilde{A}(z)F_1(z,t,\tau) \\ F_1(z,t,t) = f(z,t) \qquad \text{for} \quad \tau = t \end{cases} \tag{8}$$

where τ is a fixed parameter that simply labels the solution and $A(z)$, $\tilde{A}(z)$, $f(z,t)$ and $g(z)$ are defined as in (5) and (6). Thus if we know the solutions of the initial value problems (7) and (8) then we also know the solution of the inhomogeneous p.d.e. (4) with the correct initial values (6), i.e.

$$F(z,t) = F_0(z,t) + \int_0^t F_1(z,t,\tau)\,d\tau \tag{9}$$

For the total probability $W(t) = \sum_{n=1}^{\infty} p_n(t)$ we have

$$\frac{\partial W}{\partial t} = -(d+\tilde{d})p_1(t) = \frac{\partial F(1,t)}{\partial t} \tag{10}$$

and hence there is a reflecting boundary at $n = 1$ when $\tilde{d}/d = -1$ and an absorbing one at $n = 0$ when $\tilde{d}/d > -1$ (the case $\tilde{d}/d < -1$ describes a negative effective death rate which we are not interested in.) For the probability $p_0(t) = 1 - \sum_{n=1}^{\infty} p_n(t)$ of the population becoming extinct at time t one gets

$$\dot{p}_0(t) = -f(1,t) = (d+\tilde{d})p_1(t) \tag{11}$$

therefore the inhomogeneous term of (4) takes into account the flux towards $n = 0$.

The initial value problems (7) and (8) are readily solved by means of the familiar method of characteristics. After lengthy but standard calculations, one finally achieves the explicit form of the solution (9)

$$F(z,t) = \left(\frac{d - bz - b(1-z)e^{(b-d)t}}{d-b}\right)^{\frac{\tilde{d}}{d}-\frac{\tilde{b}}{b}} g\left(\frac{d - bz - d(1-z)e^{(b-d)t}}{d - bz - b(1-z)e^{(b-d)t}}\right)$$
$$- \frac{d+\tilde{d}}{(d-b)^{\frac{\tilde{d}}{d}-\frac{\tilde{b}}{b}}} \int_0^t \frac{\left[d - bz - d(1-z)e^{(b-d)(t-\tau)}\right]^{\frac{\tilde{d}}{d}}}{\left[d - bz - b(1-z)e^{(b-d)(t-\tau)}\right]^{\frac{\tilde{b}}{b}}}\, p_1(\tau)\mathrm{d}\tau \tag{12}$$

Notice that the integrand in (12) converges uniformly with respect to τ and one can take a derivative with respect to z directly.

Note moreover that the solution naturally splits into two parts: the first one is important only at short times and close to the initial condition $g(z)$, the second one dominates at long time scales and close to the left boundary at $n = 0$ if $b < d$ or close to both boundaries (zero and infinity) if $b > d$. Obviously this is not at all a complete solution to our problem, because we do not know $p_1(t)$ yet. Anyway we can readily obtain an equation for $p_1(t)$ by requiring that it is constructed so as to satisfy the condition

$$\lim_{z\to 0^+} F(z,t) = 0 \tag{13}$$

that is always true in the absorbing case (i.e. $\tilde{d}/d > -1$ and $x \equiv b/d$, $0 < x < 1$ from now on, for avoiding the possibility of demographic explosion). Notice that

$\lim_{z\to 1-} F(z,t) = 1 - p_0(t)$ only yields the identity $p_0(t) = (d+\tilde{d})\int_0^t p_1(\tau)\mathrm{d}\tau$. Conversely the limit (13) immediately gives the following integral equation for $p_1(t)$

$$d_1\int_0^t \frac{\left[1-e^{(b-d)(t-\tau)}\right]^{\frac{\tilde{d}}{d}}}{\left[1-xe^{(b-d)(t-\tau)}\right]^{\frac{\tilde{b}}{b}}}\, p_1(\tau)\, d\tau = \frac{\left[1-e^{(b-d)t}\right]^{\frac{\tilde{d}}{d}+N}}{\left[1-xe^{(b-d)t}\right]^{\frac{\tilde{b}}{b}+N}} \tag{14}$$

where we have chosen the initial condition $p_n(0) = \delta_{n,N}$, which implies $g(z) = z^{\frac{\tilde{d}}{d}+N}$ due to the definition of the generating function in (3) and (6). The Eq. in (14) is a Volterra equation (see Ref. 20) whose kernel, having no singularities for $0 < x < 1$, is analytic and only depends on $t-\tau$. Therefore the unique solution is analytic and may be found by means of the familiar Laplace transform.

Thus by taking the Laplace transform of the integral Eq. (14) and using the convolution theorem (see Ref. 18), one achieves for $s > 0$

$$\mathfrak{L}\{p_1(t)\} \equiv \int_0^\infty p_1(t)e^{-st}dt \equiv \tilde{p}_1(s) = \frac{1}{d_1}\frac{\mathfrak{L}\{\chi(t)\}}{\mathfrak{L}\{\kappa(t)\}} \tag{15}$$

where

$$\kappa(t) \equiv \frac{\left[1-e^{(b-d)t}\right]^{\frac{\tilde{d}}{d}}}{\left[1-xe^{(b-d)t}\right]^{\frac{\tilde{b}}{b}}}, \quad \chi(t) \equiv \frac{\left[1-e^{(b-d)t}\right]^{\frac{\tilde{d}}{d}+N}}{\left[1-xe^{(b-d)t}\right]^{\frac{\tilde{b}}{b}+N}} \tag{16}$$

Now with the change of variable

$$e^{(b-d)t} = \frac{1-z}{1-xz} \tag{17}$$

the Laplace transform $\tilde{\kappa}(s)$ of the kernel $\kappa(t)$ may be written as

$$\tilde{\kappa}(s) = \mathcal{K}\int_0^1 z^{\frac{\tilde{d}}{d}}(1-z)^{\frac{s}{d-b}-1}(1-xz)^{\frac{\tilde{b}}{b}-\frac{\tilde{d}}{d}-\frac{s}{d-b}-1}\mathrm{d}z \tag{18}$$

where $\mathcal{K} = \frac{(1-x)^{\frac{\tilde{d}}{d}-\frac{\tilde{b}}{b}}}{d}$. As $s > 0$ and $x < 1$ we can use the integral representation of the standard hypergeometric function (see Refs. 21 and 22), then we may rewrite the previous equation as

$$\tilde{\kappa}(s) = \mathcal{K}\frac{\Gamma(\beta)\,\Gamma(\gamma_s-\beta)}{\Gamma(\gamma_s)}\; F(\alpha_s,\beta,\gamma_s;x) \tag{19}$$

with the following definitions

$$\begin{cases} \alpha_s \equiv \dfrac{s}{d-b} + 1 + \dfrac{\tilde{d}}{d} - \dfrac{\tilde{b}}{b} \\ \beta \equiv 1 + \dfrac{\tilde{d}}{d} \\ \gamma_s \equiv \dfrac{s}{d-b} + 1 + \dfrac{\tilde{d}}{d} \end{cases} \tag{20}$$

where $\Gamma(z)$ and $F(\alpha, \beta, \gamma; x)$ are the gamma function and the standard hypergeometric function respectively (see Refs. 21 and 22). Note that, as x is a fixed parameter smaller than one, in Eq. (19) the condition $s > 0$ can be dropped, because in the hypergeometric function the variable s appears only inside its coefficients. Therefore Eq. (19) may be regarded as the analytic continuation of the Laplace transform $\tilde{\kappa}(s)$ inside the complex plane and then s may assume negative values as well. Furthermore the function (19) is an entire function of s by virtue of the denominator $\Gamma(\gamma_s)$, that removes the simple poles of $F(\alpha, \beta, \gamma; x)$. In fact, there are also the simple poles $s_n = -n(d-b)$, owing to the gamma function, but these will be dropped later.

The preceding integral representation of the hypergeometric function enable us to carry out the analytic continuation of $\mathfrak{L}\{\chi(t)\}$ as well, removing the restrictions on the variable s. Finally, one gets for $\mathfrak{L}\{\chi(t)\}$ the following expression

$$\mathcal{K}\frac{\Gamma(\beta+N)\,\Gamma(\gamma_s-\beta)}{\Gamma(\gamma_s+N)}\;F(\alpha_s, \beta+N, \gamma_s+N; x) \tag{21}$$

The Eqs. (19) and (21) lead to

$$\tilde{p}_1(s) = \frac{\Gamma(\beta+N)}{d_1\Gamma(\beta)}\,\frac{\Gamma(\gamma_s)}{\Gamma(\gamma_s+N)}\,\frac{F(\alpha_s, \beta+N, \gamma_s+N; x)}{F(\alpha_s, \beta, \gamma_s; x)} \tag{22}$$

and is the solution in (15). It is worth noting that the integral equations of the form (14), i.e.

$$\int_0^t \mathcal{N}(t-\tau)p_1(\tau)\,\mathrm{d}\tau = \mathcal{I}(t) \tag{23}$$

with $\mathcal{I}(0) = 0$ can be solved with the aid of the residue theorem. In fact, by means of the convolution theorem, we find

$$\tilde{p}_1(s) = \frac{\tilde{\mathcal{I}}(s)}{\tilde{\mathcal{N}}(s)} \tag{24}$$

where $\tilde{\mathcal{I}}(s)$ and $\tilde{\mathcal{N}}(s)$ are the Laplace transforms of $\mathcal{I}(t)$ and $\mathcal{N}(t)$ respectively. If $\tilde{\mathcal{N}}(s)$ and $\tilde{\mathcal{I}}(s)$ are analytic functions such that $\tilde{\mathcal{N}}(s)$ has simple, isolated zeros at $s = -\mu_\ell$ but $\partial_s\tilde{\mathcal{N}}(-\mu_\ell) \neq 0$ and $\tilde{\mathcal{I}}(-\mu_\ell) \neq 0$ for any ℓ (see below for the meaning of μ_ℓ), then the solution can be written down as

$$p_1(t) = \sum_\ell \frac{\tilde{\mathcal{I}}(-\mu_\ell)}{\partial_s \tilde{\mathcal{N}}(-\mu_\ell)} e^{-\mu_\ell t} \tag{25}$$

Therefore by means of (22) and if one knows the μ_ℓ's, this produces a complete solution to the problem of finding the generating function in (12).

3. EIGENFUNCTIONS AND EIGENVALUES IN THE ABSORBING CASE

3.1. Eigenfunctions

When $b \neq d$ one can prove that linear one-step processes have a discrete spectrum with only real non-negative eigenvalues (see Ref. 16). We now seek the probability distribution in an eigenfunction expansion of the form

$$p_n(t) = P_n^{\text{stat}} + \sum_{\ell=1}^{\infty} c_\ell \phi_n^\ell e^{-\mu_\ell t} \qquad \text{for} \quad n = 0, 1, 2, \ldots \tag{26}$$

The (right) eigenfunctions are ϕ_n^ℓ (with $\phi_n^0 = P_n^{\text{stat}}$, $c_0 = 1$) and the eigenvalues are such that

$$\mu_0 = 0 < \mu_1 < \mu_2 < \ldots < \mu_\ell < \ldots \qquad \text{for} \quad \ell \in \mathbb{N} \tag{27}$$

In this form the eigenvalues μ_ℓ are always non-negative and they are the same both for right and left eigenfunctions. General orthogonality and completeness properties of birth and death eigenfunctions have been firstly studied by Ledermann and Reuter Ref. 16 and secondly by Karlin and McGregor Ref. 7 and 8. When the spectrum is discrete, it is known that in the linear case the right eigenvectors are the Meixner or the associated Meixner polynomials (see Ref. 9). These polynomials are orthogonal with respect to a discrete measure (see Appendix B, Refs. 23 and 24). Now we obtain these polynomials in a more direct way.

The previous definitions, the following relations hold in the abc case

$$F(z,t) = z^\varepsilon \sum_{n=1}^{\infty} p_n(t) z^n = \sum_{\ell=1}^{\infty} c_\ell \phi^\ell(z) e^{-\mu_\ell t}$$

$$\phi^\ell(z) \equiv z^\varepsilon \sum_{n=1}^{\infty} \phi_n^\ell z^n \qquad \text{for} \quad \ell = 1, 2, \ldots \tag{28}$$

If we carry out the Laplace transform of $p_n(t)$ and $F(z,t)$, we get ($n > 0$)

$$\tilde{p}_n(s) = \sum_{\ell=1}^{\infty} \frac{c_\ell \phi_n^\ell}{s + \mu_\ell}$$

$$\tilde{F}(z,s) = \sum_{\ell=1}^{\infty} \frac{c_\ell \phi^\ell(z)}{s+\mu_\ell} \tag{29}$$

It should be noted that $\tilde{F}(z,s)$ is well defined with respect to the variable s in the whole complex plane, and thus this equation may be defined as the analytic continuation of the Laplace transform (15) as carried out for Eq. (19).

Moreover, for our purposes it is important to stress that the (simple) poles of $\tilde{F}(z,s)$ (or $\tilde{p}_1(s)$ as well) are the eigenvalues of the expansion (26) and the ℓ-th residue of $\tilde{F}(z,s)$ is proportional to the generating function for the ℓ-th eigenfunction. Indeed if $\mathcal{C}_{\mu_\ell}$ is a closed simple path that encircles only the singularity $s=-\mu_\ell$ then, by means of the residue theorem, it turns out

$$\oint_{\mathcal{C}_{\mu_\ell}} \tilde{F}(z,s)\,\mathrm{d}s = 2\pi i\; c_\ell \phi^\ell(z). \tag{30}$$

One finds that

$$\phi^\ell(z) \propto \lim_{s\to-\mu_\ell} (s+\mu_\ell)\tilde{F}(z,s). \tag{31}$$

Owing to the equation $p_0(t) = d_1 \int_0^t p_1(\tau)\mathrm{d}\tau$ and to the expression in (26) for $n=1$, one readily gets

$$p_0(t) = 1 - d_1 \sum_{\ell=1}^{\infty} \frac{c_\ell}{\mu_\ell} \phi_1^\ell e^{-\mu_\ell t} \tag{32}$$

because $\lim_{t\to\infty} p_0(t) = 0$, whereas $p_0(t) = 1 + \sum_{\ell=1}^{\infty} c_\ell \phi_0^\ell e^{-\mu_\ell t}$ by definition in (26). Hence one obtains

$$\phi_0^0 = 1 \qquad \text{and} \qquad \phi_0^\ell = -\frac{d_1}{\mu_\ell}\phi_1^\ell \qquad \text{for} \quad \ell = 1,2,\ldots \tag{33}$$

In order to achieve a general expression for ϕ_n^ℓ, we shall exploit the relation (31). When carrying out the Laplace transform it is possible to follow two different routes. Either one finds the solution of the Laplace transformed Eq. (4) or one carries out directly the Laplace transform of the solution (12). If the integral $\int_0^\infty F(z,t)e^{-st}\mathrm{d}t$ converges uniformly with regard to z and s in their respective domains, then the two ways would be equivalent. But this is not the case. In fact, it is possible to see that in general $\partial_z \mathfrak{L}\{F(z,t)\} \neq \mathfrak{L}\{\partial_z F(z,t)\}$, which breaks the uniform convergence of the previous integral (see Ref. 25).

Laplace transforming the Eq. (4), one can see that the solution $\tilde{F}(z,s)$ has the form

$$\tilde{F}(z,s) = C(z,s) + D(z,s)\tilde{p}_1(s) \tag{34}$$

where

$$
\begin{aligned}
C(z,s) &= -\frac{1}{d} z^{\beta+N}(1-z)^{\beta-\gamma_s}(1-xz)^{\alpha_s-1} \\
&\quad \times \int_0^1 \mathrm{d}t\; t^{\beta-1+N}(1-zt)^{\gamma_s-\beta-1}(1-xzt)^{-\alpha_s} \\
D(z,s) &= \beta z^{\beta}(1-z)^{\beta-\gamma_s}(1-xz)^{\alpha_s-1} \\
&\quad \times \int_0^1 \mathrm{d}t\; t^{\beta-1}(1-zt)^{\gamma_s-\beta-1}(1-xzt)^{-\alpha_s}
\end{aligned}
\tag{35}
$$

But the Laplace transform of the solution (12) has the form

$$
\tilde{\mathcal{F}}(z,s) = \mathcal{C}(z,s) + \mathcal{D}(z,s)\tilde{p}_1(s) \tag{36}
$$

where

$$
\begin{aligned}
\mathcal{C}(z,s) &= C(z,s) + \frac{1}{d}(1-z)^{\beta-\gamma_s}(1-xz)^{\alpha_s-1} \\
&\quad \times \frac{\Gamma(\beta+N)\,\Gamma(\gamma_s-\beta)}{\Gamma(\gamma_s+N)}\; F(\alpha_s,\beta+N,\gamma_s+N;x) \\
\mathcal{D}(z,s) &= D(z,s) - \beta(1-z)^{\beta-\gamma_s}(1-xz)^{\alpha_s-1} \\
&\quad \times \frac{\Gamma(\beta)\,\Gamma(\gamma_s-\beta)}{\Gamma(\gamma_s)}\; F(\alpha_s,\beta,\gamma_s;x)\,.
\end{aligned}
\tag{37}
$$

It is worth noting that when $F(\alpha_s, \beta, \gamma_s; x) = 0$ (as a function of s), one obtains in general $\mathcal{D}(z, -\mu_\ell) = D(z, -\mu_\ell)$ but $\mathcal{C}(z, -\mu_\ell) \neq C(z, -\mu_\ell)$, regardless of z. As it will be pointed out below, this ensures us that if $s = -\mu_\ell$, the failure of the uniform convergence affects only the part of (12) involving the initial conditions. Therefore the two preceding routes are equivalent in order to obtain eigenvalues and eigenvectors, even though the correct generating function of the process to be used is (12).

The functions $C(z, s)$ and $D(z, s)$ do not have any poles (see Appendix A) for $s = -\mu_\ell$ (note that if $C(z, s)$ had any poles for $s = -\mu_\ell$ then the eigenvalues would depend on the initial conditions). Thus we find the generating function of the right eigenfunctions, which is that of the associated Meixner polynomials (as in Ref. 9)

$$
\phi^{\ell}(z) \propto D(z, -\mu_\ell) \tag{38}
$$

Now one may succeed in drawing out the eigenfunctions ϕ_n^ℓ by expanding $D(z, -\mu_\ell)$ in powers of z. Such an expansion is readily accomplished by making use of a formula which is proven in Appendix B. The calculations (made in Appendix A) lead to the eigenfunctions ϕ_n^ℓ (for $n, \ell = 1, 2, \ldots$), which are proportional to

$$\sum_{k=0}^{n-1} \frac{1}{k+\beta} P_k^{(\varrho\,,\,\sigma_\ell-1-k)}(2x-1) P_{n-1-k}^{(-\varrho\,,\,n-1-k-\sigma_\ell)}(2x-1) \tag{39}$$

where $P_n^{(\alpha,\beta)}(z)$ are the Jacobi's polynomials (see Refs. 9, 21 and 26) and

$$\begin{aligned} \varrho &= 1 + \frac{\tilde{d}}{d} - \frac{\tilde{b}}{b} \\ \sigma_\ell &= \frac{-\mu_\ell}{d-b} \end{aligned} \tag{40}$$

Thus, if we are given the eigenvalues μ_ℓ then we have solved the problem of finding the time dependent solution in (26) for the abc case.

The previous considerations can be put forward again for the left eigenvectors. After setting $\varepsilon = \tilde{b}/b - 1 > -1$, it is possible to see that the solution of the Laplace transformed equation (using now left eigenvectors) has a form similar to (34), that is

$$\tilde{G}(z,s) = \bar{C}(z,s) + \bar{D}(z,s)\tilde{q}_1(s) \tag{41}$$

but in this case we have

$$\begin{aligned} \bar{D}(z,s) = {} & \bar{\beta} z^{\bar{\beta}} (1-z)^{\bar{\beta}-\bar{\gamma}_s} \left(1 - \frac{z}{x}\right)^{\bar{\alpha}_s - 1} \\ & \times \int_0^1 \mathrm{d}t \; t^{\bar{\beta}-1} (1-zt)^{\bar{\gamma}_s - \bar{\beta} - 1} \left(1 - \frac{z}{x} t\right)^{-\bar{\alpha}_s} \end{aligned} \tag{42}$$

where

$$\begin{cases} \bar{\alpha}_s \equiv -\dfrac{s}{d-b} - \dfrac{\tilde{d}}{d} + \dfrac{\tilde{b}}{b} \\ \bar{\beta} \equiv \dfrac{\tilde{b}}{b} \\ \bar{\gamma}_s \equiv -\dfrac{s}{d-b} + 1 + \dfrac{\tilde{b}}{b} \end{cases} \tag{43}$$

Thus the generating function of the left eigenfunctions is $\bar{\phi}^\ell(z) \propto \bar{D}(z, -\mu_\ell)$ and one may succeed in drawing out the left eigenfunctions $\bar{\phi}_n^\ell$ by expanding $\bar{D}(z, -\mu_\ell)$ in powers of z as just seen. The relations between right and left coefficients are

$$\begin{cases} \bar{\alpha}_s = 1 - \alpha_s \\ \bar{\beta} = \gamma_s - \alpha_s \\ \bar{\gamma}_s = 1 - \alpha_s + \beta \end{cases} \tag{44}$$

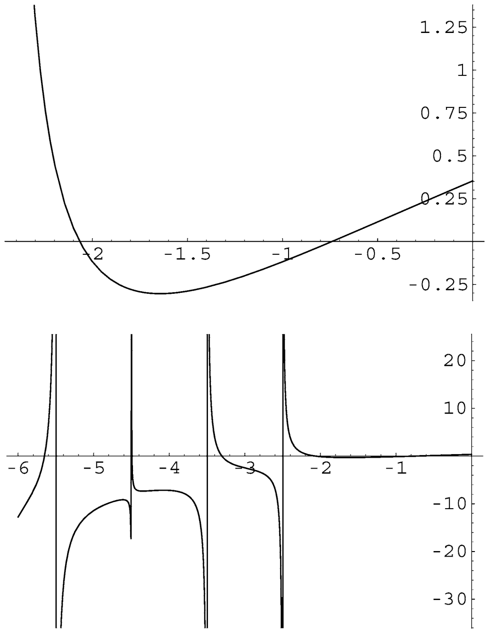

Fig. 1. The upper figure shows the first two roots of $F(\alpha_s, \beta, \gamma_s; x)$ as a function of s when $b = 1$, $d = 2$, $\tilde{b} = 4$, $\tilde{d} = 3$. For the same parameters the lower figure shows the first five roots.

3.2. Eigenvalues

When $\tilde{b} \neq 0$ it is easy to show that the only poles of $\tilde{p}_1(s)$ are the solutions of the equation

$$F(\alpha_s, \beta, \gamma_s; x) = 0 \tag{45}$$

as a function of s. Otherwise, if $\tilde{b} = 0$, the only poles of $\tilde{p}_1(s)$ are those of $\Gamma(\gamma_s)$.

In fact, when $\tilde{b} \neq 0$ the function $F(\alpha_s, \beta, \gamma_s; x)$ does depend on s and the product

$$\frac{1}{\Gamma(\gamma)} F(\alpha, \beta, \gamma; x) \tag{46}$$

is an entire function of α, β and γ, for fixed x (see Ref. 22). Furthermore, when $N > 0$ there are no zeros of $F(\alpha_s, \beta + N, \gamma_s + N; x)$ that overlap those of $F(\alpha_s, \beta, \gamma_s; x)$, otherwise the eigenvalues would depend on the initial conditions. On the contrary, if $\tilde{b} = 0$ then $F(\alpha_s, \beta, \gamma_s; x)$ does not depend on s and we have only the poles of $\Gamma(\gamma_s)$.

It is not possible to give an analytical expression for the eigenvalues in full generality (Fig. 1 shows the plot of $F(\alpha_s, \beta, \gamma_s; x)$ as a function of s for arbitrary values of $b, d, \tilde{b}, \tilde{d}$). Yet when $x \to 1$, it is possible to get a simpler expression for μ_1. If the parameters of the standard hypergeometric function $F(a, b, c; z)$ are such that $\Re(c - a - b) > 0$, then it is possible to calculate the value $F(a, b, c; 1)$ (see Ref. 22):

$$\lim_{z \to 1^-} F(a, b, c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \tag{47}$$

where $\Gamma(z)$ is the gamma function. If the first eigenvalue is much less than 1 in units of $d - b$, then $|s/(d - b)| \ll 1$ and we can write

$$\mu_1 = 2(d - b)\left|\frac{\partial_{\varsigma=0} F(\alpha_s, \beta, \gamma_s; 1)}{\partial^2_{\varsigma=0} F(\alpha_s, \beta, \gamma_s; 1)}\right| \tag{48}$$

where $\varsigma = s/(d - b)$. Thus, when $\tilde{b} > \beta b$ it is possible to write down at the leading order

$$\mu_1 = (d - b)\frac{1}{|\Gamma(\beta) + \bar{\gamma}|} \tag{49}$$

where $\Gamma(z)$ is the logarithmic derivative of the gamma function (see Ref. 22) and $\bar{\gamma} = 0.577\ldots$ is the Euler's constant. In order to get $\mu_1 \ll 1$, one also needs that $1 \ll \beta < \tilde{b}/b$.

When the parameter $\tilde{b}$ is zero, the function $F(\alpha_s, \beta, \gamma_s; x)$ does not depend on s because $\alpha_s = \gamma_s$; in this case $F(\alpha_s, \beta, \alpha_s; x) = (1 - x)^{-\beta}$. As the poles of $\Gamma(z)$ are in $z = -n$ ($n = 0, 1, 2\ldots$), we immediately deduce the eigenvalues to be

$$\mu_\ell = (d - b)(\ell + \frac{\tilde{d}}{d}) \qquad \text{for} \quad \ell = 1, 2, \ldots \tag{50}$$

4. A MORE GENERAL CASE

It is possible to generalize the preceding birth-death master Eq. (1) with the following one:

$$\begin{cases} \dfrac{\partial p_n(t)}{\partial t} = b_{n-1}p_{n-1}(t) + d_{n+1}p_{n+1}(t) - (b_n + d_n)p_n(t) \\[2ex] \dfrac{\partial p_1(t)}{\partial t} = d_2 p_2(t) - (b_1 + d_1\eta)p_1(t) \end{cases} \tag{51}$$

in which the first equation holds for $n = 2, 3, \ldots$, here $\dot{p}_0(t) = d_1\eta p_1(t)$ and the other birth and death rates are defined as before ($b_1 = b + \tilde{b} > 0$, $d_1 = d + \tilde{d} > 0$). Even in this case our master equation involves artificial boundaries, but now the pure real parameter η (≥ 0) will enable us to provide a solution which embodies at the same time the *absorbing (abc) and reflecting (rbc) boundary conditions* with a slight additional effort with respect to the previous sections. In fact, by using the total probability $W(t) = \sum_{n=1}^{\infty} p_n(t)$, we have

$$\frac{\partial W}{\partial t} = -(d + \tilde{d})\eta p_1(t) \tag{52}$$

hence there is a reflecting boundary at $n = 1$ when $\eta = 0$ and an absorbing one at $n = 0$ when $\eta \neq 0$. If $\eta = 1$, we recover the regular absorbing boundary studied yet. When $\eta \neq 0$, it is clear that the stationary solution is always $P_n^{\text{stat}} = \delta_{n,0}$, whereas if $\eta = 0$ we have a non trivial stationary solution which is

$$P_n = \mathcal{N} \frac{x^n}{n + \delta} \frac{(\beta)_n}{(\delta)_n} \qquad n = 1, 2, \ldots \tag{53}$$

where $x = b/d$, $0 < x < 1$ and $\mathcal{N}$ is a constant; $(\cdot)_n$ is the Pochhammer symbol, i.e. $(a)_0 = 1, (a)_n = a(a+1)(a+2)\ldots(a+n-1)$ for $n = 1, 2, \ldots$ (see Ref. 26 for further properties); eventually $\beta = \tilde{b}/b > 0$, $\delta = \tilde{d}/d > 0$.

Now by making similar calculations and suitable adjustments with respect to the previous two sections, it is possible to find a solution to our problem. The generating function $F(z, t) = z^{\varepsilon} \sum_{n=1}^{\infty} p_n(t) z^n$, in which $\varepsilon = \tilde{d}/d$, now leads to a first order p.d.e. for $F(z, t)$ as in (4) but with a slightly different inhomogeneous term $f(z, t) = -d_1 z^{\frac{\tilde{d}}{d}} [1 + (\eta - 1)z] p_1(t)$. The condition (13) produces a more general integral equation for $p_1(t)$ than (14). By taking the Laplace transform of this integral equation and using the same notation as in (20), one attains the generalization of (22), namely

$$\tilde{p}_1(s) = \frac{1}{d + \tilde{d}} \frac{\dfrac{\Gamma(\beta + N)}{\Gamma(\gamma_s + N)} F(\alpha_s, \beta + N, \gamma_s + N; x)}{\dfrac{\Gamma(\beta)}{\Gamma(\gamma_s)} F(\alpha_s, \beta, \gamma_s; x) + (\eta - 1) \dfrac{\Gamma(\beta + 1)}{\Gamma(\gamma_s + 1)} F(\alpha_s, \beta + 1, \gamma_s + 1; x)} \tag{54}$$

Now one may succeed in deducing the eigenvalues by making use of the recursion relations for the standard hypergeometric functions (see Ref. 26). The eigenvalues are the solutions of the equation

$$(\beta - \gamma_s) F(\alpha_s, \beta, \gamma_s + 1; x) = \eta \beta F(\alpha_s, \beta + 1, \gamma_s + 1; x) \tag{55}$$

as a function of s for fixed η and other parameters. In the rbc case we find

$$(\beta - \gamma_s) F(\alpha_s, \beta, \gamma_s + 1; x) = 0 \tag{56}$$

where the first factor simply tells us that the rbc case has the solution $s = 0$, namely the eigenvalue which corresponds to the non trivial steady-state solution. The other factor provides us nonvanishing solutions unless $b_1 = 0$ (Fig. 2).

Even in this case, when $x \to 1$ it is possible to get a simpler expression for μ_1. If the first non trivial eigenvalue is much less than 1 in units of $d - b$, then

$|s/(d-b)| \ll 1$ and now we can write

$$\mu_1 = (d-b)\left|\frac{F(\alpha_0,\beta,\gamma_0+1;1)}{\partial_{\varsigma=0}F(\alpha_s,\beta,\gamma_s+1;1)}\right| \tag{57}$$

where $\varsigma = s/(d-b)$. As before we can exploit the value of $F(a,b,c;1)$ to obtain at leading order

$$\mu_1 = (d-b)\frac{1}{|\Gamma(\beta+1)+\bar{\gamma}|} \tag{58}$$

where $\Gamma(z)$ is the logarithmic derivative of the gamma function (see Ref. 22) and $\bar{\gamma} = 0.577\ldots$ is the Euler's constant. As $\mu_1 \ll 1$ one also needs that $1 \ll \beta < \tilde{b}/b$.

When the time is about μ_1^{-1}, the solution in (26) is dominated by the first eigenfunction. In general this latter is different in the reflecting and absorbing boundaries and the eigenvalues are distinct as well. Anyway, if $\mu_1^{(\mathrm{ref})}$ ($\mu_1^{(abs)}$) is the first eigenvalue for the reflecting (absorbing) case, it is possible to see that $\mu_1^{(\mathrm{ref})} < \mu_1^{(abs)}$ for $\beta \gg 1$ (by using the Eqs. (49) and (58)). Therefore within this regime, the absorbing stationary solution is reached more rapidly than the reflecting one.

In order to derive the eigenfunctions ϕ_n^ℓ, we can proceed as above. Even in this case the previous remarks about the uniform convergence of $\int_0^\infty F(z,t)e^{-st}\mathrm{d}t$ hold. The solution of the Laplace transformed equation now has the form

$$\tilde{F}(z,s) = C(z,s) + [D(z,s) + (\eta-1)E(z,s)]\tilde{p}_1(s) \tag{59}$$

where $C(z,s)$ and $D(z,s)$ are as above and

$$E(z,s) = \beta z^{\beta+1}(1-z)^{\beta-\gamma_s}(1-xz)^{\alpha_s-1} \times \int_0^1 \mathrm{d}t\; t^\beta (1-zt)^{\gamma_s-\beta-1}(1-xzt)^{-\alpha_s} \tag{60}$$

Now we may obtain the eigenfunctions by expanding $D(z,s)+(\eta-1)E(z,s)$ in powers of z. Such an expansion produces for $n,\ell = 1,2,\ldots$

$$\phi_n^\ell = \mathcal{N}\sum_{k=0}^{n-1}\left[\frac{1}{k+\beta} + (\eta-1)\frac{1-\delta_{n,1}}{k+\beta+1}\right] \times P_k^{(\varrho\,,\,\sigma_\ell-1-k)}(2x-1)P_{n-1-k}^{(-\varrho\,,\,n-1-k-\sigma_\ell)}(2x-1) \tag{61}$$

where $\mathcal{N}$ is a constant and the other quantities are defined as in (39).

If we use left eigenvectors, then we gain for the solution of the Laplace transformed equation

$$\tilde{G}(z,s) = \bar{C}(z,s) + [\bar{D}(z,s) + (\eta-1)\bar{E}(z,s)]\tilde{q}_1(s) \tag{62}$$

where the new function is

$$\bar{E}(z,s) = \frac{d+\tilde{d}}{b} z^{\bar{\beta}+1}(1-z)^{\bar{\beta}-\bar{\gamma}_s}(1-\frac{z}{x})^{\bar{\alpha}_s-1} \times \int_0^1 \mathrm{d}t\; t^{\bar{\beta}}(1-zt)^{\bar{\gamma}_s-\bar{\beta}-1}(1-\frac{z}{x}t)^{-\bar{\alpha}_s} \tag{63}$$

and the coefficients are defined as in (43).

5. CONCLUSIONS

In this paper we have obtained a spectral resolution of a linear birth-death process by exploiting the properties of the solution of the Laplace transformed equation of the process. Our self-contained approach allowed us to obtain the eigenvectors and the relative eigenvalues of a more general problem (which includes both absorbing and reflecting boundaries).

One of the main applications of this spectral resolution concerns population dynamics. When studying ecosystems, it is possible to take into account an asymmetry between rare and common species by introducing linear birth and death rates, i.e. $b_n = bn + \tilde{b}$ and $d_n = dn + \tilde{d}$ respectively (for an arbitrary species with n individuals). If there is no immigration or speciation, one should solve the master equation with an artificial absorbing boundary at $n = 0$. It is important to stress that, even though the stationary solution is trivial, the first eigenfunction of $p_n(t)$ can still be interpreted as the population distribution of a given species on time scales μ_1^{-1}, where μ_1 is the first eigenvalue. Two major simplifications of our analysis are the non-interacting ideal gas like assumption and ignoring the effects of the spatial distribution. It would be interesting to probe what qualitative changes arise on going beyond the mean field-like theory presented here.

APPENDIX A: EIGENFUNCTIONS

In this Appendix we wish to expand the function $D(z,s)$ defined in (35) to obtain the eigenfunctions of $p_n(t)$. By using the identity obtained in (B2), we may at once write the integral inside $D(z,s)$ (because of the uniform convergence of the series) in the following form

$$\int_0^1 \mathrm{d}t\; t^{\beta-1}(1-zt)^{\alpha_s-\sigma}(1-xzt)^{-\alpha_s}$$
$$= \sum_{i=0}^{\infty} \frac{(\sigma)_i}{i!} F(-i,\alpha_s,\sigma;1-x) z^i \int_0^1 t^{i+\beta-1}\mathrm{d}t$$
$$= \sum_{i=0}^{\infty} \frac{(\sigma)_i}{i+\beta} F(-i,\alpha_s,\sigma;1-x)\frac{z^i}{i!}$$

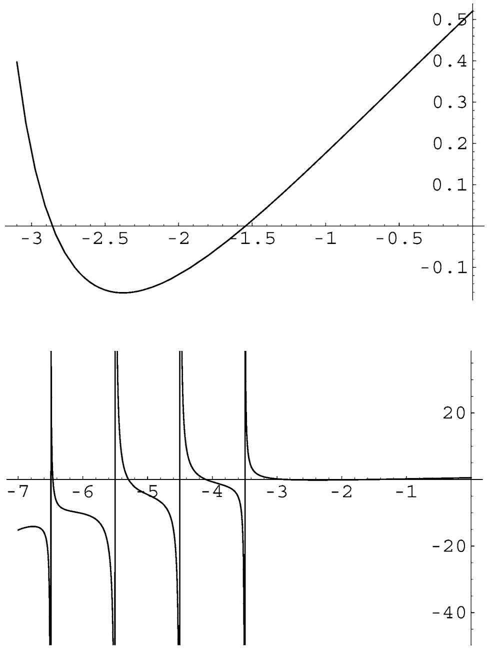

Fig. 2. The upper figure shows the first two nontrivial roots of $F(\alpha_s, \beta, \gamma_s + 1; x)$ as a function of s when $b = 1, d = 2, \tilde{b} = 4, \tilde{d} = 3$ in the rbc case. For the same parameters the lower figure shows the first five nontrivial roots.

where we have defined $\sigma \equiv 2 + \tilde{d}/d - \tilde{b}/b$; $\beta > 0$, α_s as in (20). The other product is

$$(1-z)^{\beta-\gamma_s}(1-xz)^{\alpha_s-1} = \sum_{m=0}^{\infty} \frac{(\tau)_m}{m!} F(-m, 1-\alpha_s, \tau; 1-x) z^m$$

with $\tau \equiv \tilde{b}/b - \tilde{d}/d$. As $\phi^\ell(z) \propto D(z, -\mu_\ell)$, then $\phi^\ell(z)$ is proportional to

$$z^{\tilde{d}/d} \sum_{n=1}^{\infty} z^n \sum_{m,i=0}^{\infty} \frac{(\tau)_m}{m!} \frac{(\sigma)_i}{i!(i+\beta)} F(-i, \alpha_\ell, \sigma; 1-x) \times F(-m, 1-\alpha_\ell, \tau; 1-x) \delta_{m+i+1,n}$$

$$= z^{\tilde{d}/d} \sum_{n=1}^{\infty} z^n \sum_{i=0}^{n-1} \frac{(\sigma)_i}{i!(i+\beta)} F(-i, \alpha_\ell, \sigma; 1-x)$$

$$\times \frac{(\tau)_{n-1-i}}{(n-1-i)!} F(i+1-n, 1-\alpha_\ell, \tau; 1-x)$$

where α_ℓ is α_s in which we have set $s = -\mu_\ell$. Thus, owing to the definitions in (28) the following relation holds when $n, \ell = 1, 2, \ldots$

$$\phi_n^\ell \propto \sum_{i=0}^{n-1} \frac{(\sigma)_i}{i!(i+\beta)} F(-i, \alpha_\ell, \sigma; 1-x) \times \frac{(\tau)_{n-1-i}}{(n-1-i)!} F(i+1-n, 1-\alpha_\ell, \tau; 1-x) \tag{A1}$$

Now we may simplify this formula by making use of the definition of the Jacobi's polynomials in (B3) of the Appendix B. Finally one achieves

$$\phi_n^\ell \propto \sum_{i=0}^{n-1} \frac{1}{i+\beta} P_i^{(1-\tau\ ,\ \alpha_\ell - i + \tau - 2)}(2x-1) \times P_{n-1-i}^{(\tau-1\ ,\ n-i-\tau-\alpha_\ell)}(2x-1)$$

which is our desired result.

APPENDIX B: GENERATING FUNCTION OF THE HYPERGEOMETRIC POLYNOMIALS

It is well-known that one can achieve an integral representation (see Ref. 22) of the hypergeometric series that is defined not only for $|z| < 1$, but is analytic in the whole complex plane excluding the z-plane cut along the real segment $[1, \infty]$. It can be written as

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \times \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} \mathrm{d}t \tag{B1}$$

if we assume that $\Re(\gamma) > \Re(\beta) > 0$ and $|\arg(1-z)| < \pi$. If either α or β is zero or a negative integer, the hypergeometric series is a polynomial in z and then the representation (B1) is no longer a multivalued function. When $\alpha = -n$ or $\beta = -n$ the series is a polynomial of degree n: these are the *hypergeometric polynomials*. In this case, if we multiply such a polynomial by $\frac{(\gamma)_n}{n!} y^n$ and sum over n, we get

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} F(-n, \beta, \gamma; z) y^n = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}$$

$$\times \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}((1-zt)y)^n \mathrm{d}t$$

with $y \in \mathbb{R}$ under the temporary assumption that the series converges. Due to the uniform convergence of the argument, it is justified to reverse the order of summation and integration. Noting that

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!}((1-zt)y)^n = F(\gamma, 1, 1; (1-zt)y) = (1-y+tzy)^{-\gamma}$$

we can write

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} F(-n, \beta, \gamma; z) y^n = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \times \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-y+tzy)^{-\gamma} \mathrm{d}t$$

Under the transformation $\zeta = \frac{zy}{y-1}$, the r.h.s. of the previous equation is equal to

$$(1-y)^{-\gamma} \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-\zeta t)^{-\gamma} \mathrm{d}t$$
$$= (1-y)^{-\gamma} F(\gamma, \beta, \gamma; \frac{zy}{y-1})$$

Hence for $|y| < \min\{1, \frac{1}{|z-1|}\}$

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} F(-n, \beta, \gamma; z) y^n = (1-y)^{\beta-\gamma}(1-y+zy)^{-\beta} \tag{B2}$$

when $\gamma \neq 0, -1, -2, \ldots$. Thus we have achieved the generating function of the hypergeometric polynomials. It is interesting to note that these polynomials are not orthogonal with respect to the z variable. If we write the hypergeometric equation in the self-adjoint form, one can readily prove that the orthogonal polynomials in the real interval [0, 1] with respect to the weight function $z^{\gamma-1}(1-z)^{\delta-\gamma-1}$ are $F(-n, n+\delta-1, \gamma; z)$ (provided that z, δ, γ are real and $\delta > \gamma > 0$). If we set $\gamma = \alpha + 1$ and $\delta = \alpha + \beta + 2$, it is possible to identify these polynomials with those of Jacobi (see Ref. 21):

$$F(-n, n+\alpha+\beta+1, \alpha+1; z) = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(1-2z) \tag{B3}$$

Nevertheless if we set $\beta = -i$ ($i \in \mathbb{N}$) in (B2) and we multiply such an equation by $\frac{(\gamma)_i}{i!}(\frac{1}{1-z})^i F(-m, -i, \gamma; z)$ and sum over i, we get

$$\sum_{i,n=0}^{\infty} \frac{(\gamma)_i}{i!} \frac{F(-m,-i,\gamma;z)}{(1-z)^i} \frac{(\gamma)_n}{n!} \frac{F(-n,-i,\gamma;z)}{(1-z)^n} = \left[\frac{z-1}{z}\right]^{\gamma}$$

after putting $y = 1/(1-z)$. This leads to the orthogonality relation

$$\sum_{i=0}^{\infty} \frac{(\gamma)_i}{i!} \frac{1}{(1-z)^i} F(-m,-i,\gamma;z) F(-n,-i,\gamma;z) = \delta_{m,n} \frac{(1-z)^n n!}{(\gamma)_n} \left(\frac{z-1}{z}\right)^{\gamma} \tag{B4}$$

We may define $M_n(i;\gamma,x) \equiv (\gamma)_n F(-n,-i,\gamma;1-\frac{1}{x})$ and finally we achieve (here $x^{-1} = 1-z$)

$$\sum_{i=0}^{\infty} M_m(i;\gamma,x) M_n(i;\gamma,x) \frac{(\gamma)_i}{i!} x^i = \delta_{m,n} \frac{n!\,(1-x)^{-\gamma}}{(\gamma)_n x^n} \tag{B5}$$

when we fix $0 < x < 1$ and $\gamma > 0$, $M_n(i;\gamma,x)$ are the Meixner polynomials, which are orthogonal with respect to the discrete measure $\frac{(\gamma)_i}{i!} x^i$.

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